Swami Ramanand TeerthMarathwada University, Nanded, Maharashtra, India

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STUDY OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEM

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Submitted by

Mr. Sukalwad Umesh Ramrao

N.E.S. Science College, Sneh Nagar,
Nanded Dist. Nanded (M.S.), India
Email:usukalwad@gmail.com

Under the supervision of

Dr. R.N. Ingle

M.sc, Ph.D
Professor and Head, Department of Mathematics,
Bahirji Smarak Mahavidyalaya, Basmathnagar,
Dist. Hingoli (M.S.), India
Email: ingleraju11@gmail.com

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I. Introduction to proposed research topic:

Fractional differential equations (DEs) have received considerable attention in the recent years due to their wide applications in the areas of applied mathematics, physics, engineering, economy, and other fields. Many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are well described by fractional DE . It is well known that the fractional order differential and integral operators are non-local operators. This is one reason why fractional differential operators provide an excellent instrument for description of memory and hereditary properties of various physical processes. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models.Motivated by increasing number of applications of fractional DEs, considerable attention has been given to provide efficient methods for exact and numerical solutions of fractional DEs. In general, most of fractional DEs do not have exact solutions. Particularly, there is no known method for solving fractional boundary value problems (BVPs) exactly. As a result, numerical and analytical techniques have been used to study such problems. It should be noted that much of the work published to date concerning exact and numerical solutions is devoted to the initial value problems for fractional order ordinary DEs. The theory of BVPs for fractional DEs has received attention quiet recently. The attention drawn tothe theory of existence and uniqueness of solutions to BVPs for fractional order DEs is evident from the increased number of recent publications. Many researcher using definition of fractional derivative and some different methods for solving to Fractional Differential Equation:

Definition of the Riemann Liouville Fractional Integral:

The Riemann–Liouville definition. For α ∈[n−1,n), the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

Definition of the Caputo's Fractional Derivative :

Caputo definition. For α ∈ [n−1,n), the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\int_a^{\alpha} \frac{f^{(n)}(x)}{(\mathrm{t}-\mathrm{x})^{\alpha-\mathrm{n}+1}}dx$$

Some Important Methods.

1) Homotopy Analysis Method

A homotopy between two continuous function $f(x)$and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $\mathcal{H}: X * [0,1] \rightarrow Y$ from the product of the space X with the unit interval [0,1] to Y such that , it x in X then $H(x;0) = f(x)$ and $\mathcal{H}(x;1) = g(x)$.

2) Novel Numerical Method:

Suppose the two dimensional Laplacian has a complete set of orthonormal eigen functions $\varphi_{n,m}$ corresponding to eigen values $\lambda_{n,m}$ in a rectangular region $D = \{(x,y)|0 \leq x \leq a, 0 \leq y \leq b\}$, $B(\varphi) = 0$ on ∂D, where $B(\varphi)$ is the homogeneous dirichlet boundary condition.

3) Fractional differential transform method:

There are several approaches to the generalization of the notion of differentiation to fractional orders. The fractional differentiation in Riemann–Liouville sense is defined by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

For $m-1 \leq q < m, m \in Z^+, x > x_0$ let us expand the analytical and continuous function f(x) in terms of a fractional power series as follows

$$f(x) = \sum_{k=0}^{\infty} F(x)(x-x_0)^{k/\alpha}$$

Where α is the order of fraction and F(k) is the fractional differential transform of f(x). concerning the practical applications encounterd in various branches of science, the fractional initial conditions are frequently not available, and it may not be clear what their physical meaning is. Therefore, the definition in above equation should be modified to deal with integer ordered initial conditions in Caputo sense as follows:

$$D_{x_0}^q\left[f(x) - \sum_{k=0}^{\infty}\frac{1}{k!}(x-x_0)^{\frac{k}{\alpha}}f^k(0)\right] = \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\left\{\int_0^x\left[\sum_{k=0}^{\infty}\frac{1}{k!}(x-x_0)^{\frac{k}{\alpha}}f^k(0)\right]dt\right\}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(k) = \begin{cases} \text{If} \frac{k}{\alpha} \in Z^+ \; then \; \frac{1}{(k/\alpha)} \left[\frac{d^{\frac{k}{\alpha}} f(x)}{dt^{\frac{k}{\alpha}}} \right]_{x=x_0}, \; \text{for} k = 0,1,2, \ldots, (n\alpha - 1) \\ \text{If} \frac{k}{\alpha} \; not \; in \; Z^+ \; then \;\; 0 \end{cases}$$

where, n is the order of FDE considered

For example, we will solve the following types of fractional partial differential equations

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \le x \le L, 0 < \alpha \le 1, t > 0$$

Initial condition: u(x,0) = $f(x), 0 \le x \le L$

Boundary condition: $u(0,t) = g_{(1),}\, u(L,t) = g_2(t), t > 0$

4) The Legendre Pseudo spectral Method: The Legendre Pseudo spectral method is a way of writing any problem modeled with differential equations as a problem with only algebraic constraints by use of the 4 Pseudo spectral differential matrix.

Purpose of carrying out proposed research work:

In recent years, the boundary value problems of fractional order differential equations have emerged as an important area of research, since these problems have applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer theology, regular variation in thermodynamics, biophysics, aerodynamics, visco elasticity and damping, electrodynamics of complex medium, wave propagation, and blood flow phenomena. The purpose of the research will develop different methods for solving fractional boundary value problem to obtain their solutions. Also, we will represent graphically. The solution of fractional boundary value problem obtains by new iterative methods using mathematical software Mathematica.

Also we will study Analytical and Numerical methods to obtain the solutions of fractional boundary value problem, such as Homotopy Analysis Method**,** Novel Numerical Method, Fractional differential transform method, Adomian Method, etc.

II. Review of literature and development in the relevant subject

1. **Bashir Ahmad, Juan J. Nieto and Johnatan Pimentel**[1]: In this paper, the authors study the existence of solutions for nonlinear fractional differential

equations and inclusions of order $q \in (1, 2]$ with families of mixed and closed boundary conditions. In case of inclusion problems, the existence results are established for convex as well as nonconvex multivalued maps. The authors results are based on Leray–Schauder degree theory, nonlinear alternative of Leray–Schauder type, and some fixed point theorems for multivalued maps. Some interesting special cases are also discussed.

2. **Ahmad El-Ajou , Omar Abu Arqub and Shaher Momani[2]:** In this paper, the homotopy analysis method is applied to provide approximate solutions for linear and nonlinear two-point boundary value problems of fractional order. The solution was calculated in the form of a convergent power series with easily computable components. In this method, one has great freedom to select auxiliary functions, operators, and parameters in order to ensure the convergence of the approximate solution and to increase both the rate and region of convergence. Numerical examples are provided to demonstrate the accuracy and efficiency of the present method. Meanwhile, further iterations can produce more accurate results and decrease the error.
3. **Nguyen Minh Dien[3]:** In this paper, the author study the nonlinear sequential fractional boundary value problems involving generalized ψ-Caputo fractional derivatives with nonlocal boundary conditions. The author investigate the Green function and some of its properties, from which the author obtain a new Lyapunov-type inequality for our problem. A lower bound for the possible eigenvalues of his problem is derived. Furthermore, the author apply some properties of the Green function to obtain some existence results for his problem. It is worth mentioning that his results still work with some source functions including singularities.
4. **Shazia Sadiqand Mujeeb ur Rehman [4]:** In this paper, a numerical method is presented to solve fractional boundary value problems. In fractional calculus, the modelling of natural phenomenons is best described by fractional differential equations. So, it is important to formulate efficient and accurate numerical techniques to solve fractional differential equations. In this paper, first, the author introduce ψ-shifted Chebyshev polynomials then project these

polynomials to formulate ψ-shifted Chebyshev operational matrices. Finally, these operational matrices are used for the solution of fractional boundary value problems.

The convergence is analysed. It is observed that solution of non-integer order differential equation converges to corresponding solution of integer order differential equation. Finally, the efficiency and applicability of method is tested by comparison of the method with some other existing methods.

5. **I. Podubny [5**]: In this paper, the author presents a summary of the different definitions of fractional derivatives and show models where they appear, including simple nonlinear systems with chaos. Existing results on the solvability of classical fractional differential equations and numerical approaches are summarized. Finally, the author propose a numerical scheme to approximate the solution to linear fractional initial value problems and boundary value problems.
6. **S. Momani [6]:**In this paper, the author uses the differential transform method (DTM) to obtain analytical solutions of fractional heat- and wave-like equations with variable coefficients. The time fractional heat-like and wave-like equations with variable coefficients were obtained by replacing a first-order and a second-order time derivative by a fractional derivative of order $0 < \alpha < 2$. The approach mainly rests on the DTM which is one of the approximate methods. The method can easily be applied to many problems and is capable of reducing the size of computational work. Some examples are presented to show the efficiency and simplicity of the method.
7. **J.H. He [7]:**In this paper the author, introduce a new analytical method called the Natural Homotopy Perturbation Method (NHPM) for solving linear and the nonlinear fractional partial differential equation is introduced. The proposed analytical method is an elegant combination of a wellknown method, Homotopy Perturbation Method (HPM) and the Natural Transform Method (NTM). In this new analytical method, the fractional derivative is computed in Caputo sense and the nonlinear terms are calculated using He's polynomials. Exact solution of linear and nonlinear fractional partial differential equations are successfully obtained using the new analytical method, and the result is compared with the result of the existing methods.

8. **S. Bhalekar, V Daftardar – Gejji [8]:**In this paper, the authors apply the New Iterative Method for solving various linear and nonlinear evolution equations. The results obtained are compared with the results by existing methods.
9. **Ozisik [9]:**In this book, the author introduces new methods for the solution of linear systems of equations; new model equations, such as for quasi-one-dimensional compressible flow; algorithms for pressure-velocity coupling in laminar incompressible flows, such as the SIMPLEC method; modern methods for the solution of hyperbolic systems based on solution features; the discretization and solution of the equation of radiative transfer; phase change problems with convection in the liquid phase and with a solute dispersed in the base material; numerical grid generation in cylindrical and polar coordinates; and recent advancements in the hybrid analytical–numerical generalized integral transform technique (GITT).
10. **A.A. Kilbas, H.M. Srivastva and J.J. Trajillo [10]:** In this paper, the author reviewed the literature about the basics, standard approaches, analytical and approximate methods to the problem of fractional differential equations, while discussing about the basic properties including the rules for their compositions and the conditions for the equivalence of various definitions.
11. **R. Hilfer [11]:**In this book, the author illustrate that fractional derivatives seem to arise generally and universally, and for deep mathematical reasons. One central theme of this book is the fact that fractional derivatives arise as the infinitesimal generators of a class of translation invariant convolution semi groups. These semigroups appear universally as attractors for coarse graining procedures or scale changes. They are parametrized by a number in the unit interval corresponding to the order of the fractional derivative.
12. **V. B. L. Chaurasia and A. Srivastava [12]:**The author discuss and derive the analytical solution for the generalized time-fractional tele-graph equation. These problems are solved by taking the Laplace and Fourier transforms in variable t and x respectively. Here the author use Green function also to derive the solution of the given differential equation.

III. Research gap:

Most important thing is that many fractional differential equations and fractional boundary value problem do not have an analytical solution. Therefore, we develop the numericalmethods to obtain there solutions. Also, todiscuss the stability and convergence of the numerical methods.Also wide scope in to obtain exact solution of fractional boundary value problem by developed new different methods and it will represent graphically.

IV. Importance of the Study:

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last decade Fractional calculus has been applied to almost every field of science, engineering and mathematics.

a) International Status: Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache etc.

b) National Status: Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V.Srivastava,K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna etc.

c) Regional Status: Many researchers from Maharashtra have been actively involve in the study of fractional calculus. Like Kiran, M. Kolwankar, A.K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, K.C. Takale, J. Verma, D.B. Dhaigudeetc.

- **Scope of Study:**

The analysis of different methods for solving fractional boundary value problem. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing,

nonlinear biological systems and astrophysics. There are few different methods for obtaining the solution of fractional boundry value problem, therefore, we develop new different methods to obtain their solutions and discuss the stability and convergence of the numerical methods upto the length. We will study the powerful mathematical software Mathematica to simulate the solutions graphically. We will develop iterative and finite difference schemes to solve them. Solutions of these equations will be simulated graphically. We intend to publish the work in reputed Journals.

V. Objectives of Proposed Research Work:

1) To study in detail Fractional boundary value problem.
2) To study various methods for solving Fractional boundary value problem.
3) To develop various methods for solving Fractional boundary value problem.
4) To study analytical and numerical methods for solving Fractional boundary value problem.
5) To discuss convergence and stability of solutions obtained by developed numerical methods.
6) To stimulate the solutions of Fractional boundary value problem using mathematical software Mathematica and to discuss the nature of the solutions.
7) To study the recent applications of different methods for solving Fractional boundary value problem in a various fields of science and technology.
8) To publish research papers in reputed journals.

VI. Hypothesis:

Numerical and iterative methods are efficient tools to study fractional boundary value problem.

The proposed work is to prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering field.

Hence, in the present work, we will try to develop the new iterative techniques for fractional boundary value problem and to obtain their solution. These will represented graphically by Mathematica software.

VII. Limitations of the work:

In our research, we consider the fractional differential equations. The fractional order model are more adequate than integer-order model, because fractional derivatives and integrals enable the description of memory properties of various materials and processes. The analytical solution of most fractional differential equations cannot be obtained, and as a consequence, approximate and numerical techniques are playing important role in identifying the solution behavior of such fractional equation.

VIII. Research Methodology:

The following steps will be including in the methodology.

1) To refer the relevant reference books on fractional boundary value problem.

2) To study Mathematicasoftware.

3) To study fractional calculus.

4) To study Iterative / Numerical methods for obtaining approximate solutions of fractional boundary value problem.

5) To discuss the stability and convergence of Iterative / Numerical methods.

6) To represent Numerical solutions of fractional boundary value problem graphically by Mathematica and discuss their nature.

IX. Expected outcome:

1) We will develop different methods for solving fractional boundary value problem and their comparison.

2) We will develop numerical methods for fractional boundary value problem.

3) To find the approximate solutions of fractional boundary value problem by developing new iterative technique and it will represented graphically by Mathematicasoftware.

X. Year wise plan of work and Targets to be Achieved:

First year plan:

1) To collect the reference books on fractional differential equation and it's boundary value problem .

2) To study the reference books on fractional differential equation and it's boundary value problem.

3) To obtain the solutions of fractional boundary value problem by iterative methods.

4) To visualize solutions by graphs using a powerful software Mathematic and MATLAB.

Second year plan:

1) To attend seminars/conferences/workshops on relevant topics.

2) To study fractional calculus.

3) To study fractional differential equations and it's boundary value problem.

4) To develop various methods to obtain the solution of fractional boundary value problem.

5) To visualize the solutions by graphs using Mathematica, MATLAB and discuss the nature.

Third year plan:

1) To develop different methods for fractional boundary value problem.

2) To study applications of obtained results.

3) We will publish our work in reputed journals.

4) Submission of completed work to concerned authority.

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Sukalwad Umesh Ramrao. **Dr. R.N. Ingle**

(Research Scholar) **(Research Guide)**